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1984 J. Phys. A: Math. Gen. 17 547

(http://iopscience.iop.org/0305-4470/17/3/016)

J. Phys. A: Math. Gen. 17 (1984) 547–557. Printed in Great Britain

Perturbative solution for the generalised anharmonic oscillators†

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Received 22 July 1983

Abstract. Serious difficulties have been encountered in the past when attempting the straightforward application of the Borel method to the summation of perturbation series in quantum mechanics and quantum field theory because of an analytic continuation of the Borel transform series which is usually performed by means of conformal mapping techniques. In this paper we show that this continuation may be accomplished by combining the Borel method with the first confluent form of the ε algorithm of Wynn: for the calculation of the ground state energy eigenvalue of the generalised anharmonic oscillators, two different approaches are presented, both of which lead to accurate results.

1. Introduction

The quantum mechanical problem of the generalised anharmonic oscillators, described in the one-dimensional case by the Hamiltonian

$$H = p^2 + x^2 + \beta x^{2m} \qquad (m = 2, 3, \ldots), \tag{1.1}$$

continues to attract an appreciable amount of interest. This is due to the analogy between this model and one-dimensional quantum field theories, both of which yield divergent perturbation expansions. Furthermore, anharmonic oscillators are also directly relevant to the study of various atomic and molecular problems of quantum chemistry. Several approximation methods have consequently been used to calculate eigenvalues and eigenvectors of the Hamiltonian (1.1): among these we may recall WKB methods (Kesarwani and Varshni 1981, 1982), the Rayleigh–Ritz variational method (Graffi and Grecchi 1973, Mitra 1977, Castro *et al* 1983), Hill's determinant (Biswas *et al* 1971, 1973, Hautot and Magnus 1979, Banerjee 1976, 1978, Banerjee *et al* 1978) and, above all, perturbation techniques (Reed and Simon 1978).

In what follows we shall be chiefly interested in the Rayleigh–Schrödinger perturbation expansion in powers of the coupling constant β, for the ground state energy of the Hamiltonian (1.1). The main reason for considering perturbative solutions to the Schrödinger equation arises from the consideration that, in quantum field theory, perturbation series are often the only practical tool available for obtaining an approximate solution of the problem in hand. This is because in many cases of practical interest it is difficult, if not impossible, to calculate the matrix elements of the operators required for performing a variational or other non-perturbative calculation.

† This research has received support from MPI contributions.

0305-4470/84/030547+11\$02.25

The most widely used way to overcome the divergence of perturbative expansions is to resort to a resummation of the series using one of a few well known methods. In the case of the anharmonic oscillators, among these techniques we recall some nonlinear summation procedures, such as Padé approximants (Loeffel *et al* 1969, Simon 1970, Iafrate and Osche 1972), the repeated application of Aitken's Δ^2 process (Van Dyke 1974, Drummond 1981), continued fractions (Reid 1967, Čižek and Vrscay 1982), the generalised Euler transformation (Gunson and Ng 1972, Bhattacharyya 1982), Levin's approximants and Brezinski's ϑ algorithm (Ford and Smith 1982). On the other hand, all these methods possess a limited range of applicability: for example, it has been rigorously proved (Graffi and Grecchi 1978) that the Padé approximants do not converge to the eigenvalues of (1.1) for $m \geqslant 4$.

In most cases one may resort to another summation procedure, the Borel summability method (Reed and Simon 1978, Hardy 1949), which is linear and whose convergence to the eigenvalues of (1.1) has been demonstrated (Graffi *et al* 1970) for arbitrary m. However, the Borel summability method does not readily lend itself to straightforward numerical application because it involves an analytic continuation, and also because the strong divergence of the coefficients in the Rayleigh–Schrödinger expansion soon makes the calculations too cumbersome. To circumvent the first complication, in the early experiments with the Borel method the analytic continuation was attempted by combining Borel summation with Padé approximants (Graffi *et al* 1970, 1971): though this procedure has subsequently received considerable attention and application to quantum mechanics and field theory (Bonnier 1978, Eletskiĭ and Popov 1978a, b, Leinaas and Osnes 1980, Popov and Weinberg 1982), it does not seem *a priori* to be completely justifiable.

The aim of this paper is to apply a recently proposed (Lovitch and Marziani 1983) combination of the Borel summability method with the first confluent form of the ε algorithm of Wynn (1960a, b), to calculate the ground state of (1.1). In § 2 we briefly review the main theoretical aspects of the anharmonic oscillators and give some results about the Borel summability of the Rayleigh–Schrödinger energy series; in § 3 we survey the properties of the first confluent form of the ε algorithm and its combination with the Borel method. Finally, in § 4 we present our numerical results as well as our conclusions.

2. Generalised anharmonic oscillators

We shall consider the one-dimensional Schrödinger equation

$$H\psi(x) = E\psi(x) \tag{2.1}$$

where the Hamiltonian H is

$$H = -\mathrm{d}^2/\mathrm{d}x^2 + x^2 + \beta x^{2m} \qquad (m = 2, 3, \ldots) \tag{2.2}$$

with the conditions

$$x \in (-\infty, +\infty), \tag{2.3}$$

$$\lim_{x \to \pm\infty} \psi(x) = 0, \tag{2.4}$$

$$\beta > 0. \tag{2.5}$$

It is well known that the ground state energy of the Hamiltonian (2.2) can be calculated by means of perturbation theory (Reed and Simon 1978): in fact, anharmonic oscillators are discussed in many standard quantum mechanics textbooks (Landau and Lifshitz 1958, Davydov 1965) as a simple example of the application of perturbation theory. In this context the ground state energy E_0 of (2.2) may be expanded in powers of the coupling constant β:

$$E_0(\beta)=\sum_{0}^{\infty}{}_n\, a_n\beta^n, \tag{2.6}$$

which is known as the Rayleigh–Schrödinger series. Unfortunately, the series (2.6) is not convergent but only asymptotic (Simon 1970) to the true eigenvalue $E_0(\beta)$: in fact, it has been shown (Bender and Wu 1971, 1973) that the coefficients a_n in (2.6) diverge factorially according to the estimate

$$a_n \underset{n\to\infty}{\cong} \mathrm{O}[\Gamma(mn-n+1)]. \tag{2.7}$$

For example, in the case of the quartic anharmonic oscillator ($m=2$) it has been proved (Bender and Wu 1973) that a_n possesses the asymptotic behaviour

$$a_n \underset{n\to\infty}{\cong} [(-1)^{n+1}6^{1/2}/\pi^{3/2}](\tfrac{3}{2})^n\Gamma(n+\tfrac{1}{2}). \tag{2.8}$$

From (2.8) it is easy to see that the corresponding series (2.6) has zero radius of convergence. In addition, it suggests the use of Borel summability which consists in recovering $E_0(\beta)$ from the series (2.6) through the integral

$$\int_0^\infty \mathrm{e}^{-x}\left\{\sum_{0}^{\infty}{}_n \frac{a_n}{[(m-1)n]!}\beta^n x^{(m-1)n}\right\}\mathrm{d}x. \tag{2.9}$$

Application of (2.9) to the calculation of $E_0(\beta)$ is justified by the following arguments:

(i) series (2.6) has been proved (Graffi *et al* 1970) to be a strong asymptotic series (Reed and Simon 1978);

(ii) it therefore satisfies the conditions of the Watson–Nevanlinna theorem (Sokal 1980).

Let us consider, for simplicity, the quartic anharmonic oscillator and define the Borel transform of the series (2.6) to be the expression in braces in (2.9), i.e.

$$B(\beta x)=\sum_{0}^{\infty}{}_n \frac{a_n}{n!}\beta^n x^n. \tag{2.10}$$

Then, by virtue of the Watson–Nevanlinna theorem, it follows that:

(i) $B(x)$ is analytic in a neighbourhood of $x=0$;

(ii) $B(x)$ possesses an analytic continuation in a neighbourhood of the real semi-axis $[0,\infty)$;

(iii) the inverse Borel transform, defined to be the integral (2.9), is convergent and yields a unique formal sum for the original series (2.6).

As we have already pointed out, the application of the Borel method, through the integral (2.9), is not straightforward. In fact, in order to perform the integration over the semi-infinite interval, one needs to know the sum of the Borel transform series $B(\beta x)$ for every real value of x. But, as one can easily see from (2.8), the Borel series

(2.10) has a finite radius of convergence ρ_B: indeed, a simple calculation shows that

$$\rho_B = \lim_{n\to\infty} \left| \frac{a_n}{n!} \frac{(n+1)!}{a_{n+1}} \right| = \frac{2}{3} \tag{2.11}$$

for the quartic anharmonic oscillator. Some procedure of analytic continuation is therefore required. Besides the aforementioned combination (Graffi *et al* 1970, 1971) of the Borel method with Padé approximants (Baker 1975), other continuation techniques have been used in the past, which are all based on conformal mapping (Le Guillou and Zinn-Justin 1977, Parisi 1977, Sobelman 1979, Zinn-Justin 1981, Hirsbrunner 1982); various choices of the conformal transformation are possible (Hirsbrunner 1982) and these generally lead to quite different results, with respect to relative errors, rate of convergence and so on.

In § 3 we shall see that the combination of the first confluent form of the ε algorithm of Wynn (1960a, b) with the Borel method provides another way for analytically continuing the Borel transform series (2.10) outside its circle of convergence.

3. First confluent form of the ε algorithm

In order to evaluate the inverse Borel transform

$$E_0(\beta) = \int_0^\infty e^{-x} B(\beta x)\, dx = \lim_{t\to\infty} \int_0^t e^{-x} B(\beta x)\, dx = \lim_{t\to\infty} S(t;\beta), \tag{3.1}$$

we make use of a 'continuous prediction' method (Brezinski 1977), namely, the first confluent form of the ε algorithm of Wynn (1960a, b). This method applies to the calculation of the limit of a complex function $S(t)$, continuous and differentiable to every order, as t tends to infinity, in terms of the values of $S(t)$ and of its derivatives for a given value t_0 of t. Taking $S(t) = S(t;\beta)$ as given in (3.1), the application of the first confluent form of the ε algorithm to this function enables one to calculate

$$E_0(\beta) = \lim_{t\to\infty} S(t) \tag{3.2}$$

as the limit, as n tends to infinity, of the sequence

$$\varepsilon_{2n}(t_0) = H_{n+1}^{(0)}(t_0)/H_n^{(2)}(t_0) \qquad (n = 0, 1, \ldots), \tag{3.3}$$

where t_0 is an arbitrary value of t and $H_k^{(n)}(t)$ are Hankel functional determinants defined by

$$H_k^{(n)}(t) = \begin{vmatrix} S^{(n)}(t) & S^{(n+1)}(t) & \ldots & S^{(n+k-1)}(t) \\ & \cdots & & \\ S^{(n+k-1)}(t) & S^{(n+k)}(t) & \ldots & S^{(n+2k-2)}(t) \end{vmatrix}, \tag{3.4}$$

with the additional conditions

$$H_0^{(n)}(t) = 1, \tag{3.5}$$

$$S^{-1}(t) = 0, \tag{3.6}$$

$$S^{(0)}(t) = S(t). \tag{3.7}$$

The class of functions $S(t)$ for which $\lim_{t\to\infty} S(t)$ equals the limit of the sequence (3.3) is usually very large (Wynn 1968) and it includes all finite linear combinations of products of exponentials and polynomials. The same is also true (Wynn 1968) for integrals of such combinations. We can therefore apply the above method to the function $e^{-x}B(\beta x)$, since inside its circle of convergence the Borel series $B(\beta x)$ can be approximated to an arbitrary degree of accuracy by a partial sum. Hence the sequence of quotients in (3.3) converges to a finite limit which equals the value of the ordinary limit (3.2).

In practical applications, one can calculate the sequence of approximants $\varepsilon_{2n}(t_0)$ by means of (3.3) and (3.4): the only restriction in the choice of the arbitrary point t_0 is imposed by the fact that the Borel series (2.10) has the finite radius of convergence ρ_B, as expressed by (2.11). Thus, in order to calculate the integral

$$S(t_0)=\int_0^{t_0} e^{-x}B(\beta x)\,dx \tag{3.8}$$

and its successive derivatives $S^{(n)}(t_0)$, one has to choose t_0 such that

$$\beta t_0<\rho_B; \tag{3.9}$$

in fact, inside the circle of convergence, the Borel series $B(\beta x)$ converges uniformly with its derivatives, which have the same radius of convergence. Therefore one can recover the Borel sum (3.1) without explicitly continuing the Borel series outside its circle of convergence: no conformal mapping technique is required in this approach. The obvious drawback to this procedure, however, derives from the fact that the arbitrary parameter t_0 must be decreased correspondingly as the value of the coupling constant β increases, as one can argue from (3.9).

An alternative approach involves the use of a generalised Borel summability method in which the Borel transform series converges everywhere.

In fact, it is known (Simon 1970) that the perturbation series (2.6) for the quartic anharmonic oscillator is asymptotic uniformly in any sector

$$|\arg\beta|<\tfrac{3}{2}\pi \tag{3.10}$$

of the complex β plane. From this condition it follows (Zinn-Justin 1981) that the generalised Borel transform

$$B(z)=\sum_0^{\infty}{}_n\,\frac{a_n}{\Gamma(\alpha n+1)}\,z^n \tag{3.11}$$

is an entire function, i.e. it has an infinite radius of convergence, if the parameter α satisfies the condition

$$1<\alpha<3. \tag{3.12}$$

Therefore, if we consider, for the quartic anharmonic oscillator, the generalised Borel transform of order two, namely

$$B(z)=\sum_0^{\infty}{}_n\,\frac{a_n}{(2n)!}\,z^n, \tag{3.13}$$

then condition (3.12) and the asymptotic formula (2.8) for the Rayleigh–Schrödinger coefficients immediately show that $B(z)$ in (3.13) is an entire function. Therefore, one can recover the ground state energy from the generalised inverse Borel transform

$$E_0(\beta)=\int_0^{\infty} e^{-x}\left\{\sum_0^{\infty}{}_n \frac{a_n}{(2n)!}\beta^n x^{2n}\right\} dx. \tag{3.14}$$

However, one finds in practice that even the generalised Borel transform series (3.13) is not easily summed, in spite of its infinite radius of convergence; this is a consequence of the fact that the terms of the series are of alternating signs and their absolute values are several orders of magnitude larger than the sum of the series. Furthermore, one knows only a limited number of coefficients, so that numerical difficulties prevent a straightforward summation of the series (3.13). However, one can again make use of the first confluent form of the ε algorithm combined now with generalised Borel summation: the only difference is that we define $B(\beta x)$ as in (3.13) and then proceed with the calculations exactly as before.

Obviously, the latter technique can be readily generalised to more ıapidly diverging perturbation series, such as the case of the sextic or octic anharmonic oscillators ($m=3$ and $m=4$, respectively, in (1.1)). For instance, for the octic oscillator Hamiltonian

$$H=p^2+x^2+\beta x^8, \tag{3.15}$$

it follows from (2.7) that the coefficients diverge as $[(3n)!]$; hence, in order to have a Borel transform series which converges everywhere, we should choose

$$B(z)=\sum_0^{\infty}{}_n \frac{a_n}{(4n)!} z^n, \tag{3.16}$$

because now the condition analogous to (3.12) is

$$3<\alpha<5. \tag{3.17}$$

Subsequently, one can approximate the inverse Borel transform by means of (3.3)–(3.7), choosing for t_0 a value which allows the series (3.16) to be easily summed.

In § 4 we shall present some results obtained with the two approaches outlined above.

4. Numerical results

We shall first consider the calculation of the ground state energy of the quartic anharmonic oscillator. The first 75 coefficients of the Rayleigh–Schrödinger perturbation series (2.6) had been calculated by Bender and Wu (1969), but these were quoted only to 12 significant figures. Since our goal was to perform as accurate a calculation as possible in order to show the power of the method, we re-computed these coefficients; we followed the method of Danforth and Swenson (1972) (see also Caswell 1979), which reduces the computation to a straightforward iteration of an algebraic recurrence relation: this method has been further modified by Killingbeck (1981) to produce renormalised series directly from hypervirial recurrence relations. We evaluated a large number (145) of coefficients in the hope of extracting the maximum possible information about the eigenvalues. A large number of coefficients is also required in the summation process of the Borel transform series and of its derivatives; we used double precision arithmetic on a Cyber 7600 computer (29 figures), truncating the results to 25 significant figures.

Following the first approach outlined in § 3, we applied the first confluent form of the ε algorithm to the function $S(t)$ of (3.1), where the Borel transform series $B(\beta x)$ is given by (2.10). The successive derivatives of $S(t)$ are given by

$$S^{(n+1)}(y) = \mathrm{e}^{-y} \sum_{0}^{\infty}{}_{k} (-1)^{n-k} \binom{n}{k} \frac{\mathrm{d}^k B(y)}{\mathrm{d}y^k} \qquad (n = 0, 1, \ldots) \tag{4.1}$$

where

$$\frac{\mathrm{d}^k B(y)}{\mathrm{d}y^k} = \sum_{k}^{\infty}{}_{m} \frac{a_m}{(m-k)!} y^{m-k}. \tag{4.2}$$

Choosing an arbitrary point t_0 subject to the condition (3.9), one can then approximate the Borel sum (3.1) by means of the sequence $\varepsilon_{2n}(t_0)$ in (3.3), employing (3.4)–(3.8). However, to avoid large round-off errors in the evaluation of the determinants (for $\beta = 1$ the determinants in the numerator and denominator of (3.3), in the case of the twentieth approximant $\varepsilon_{40}(t_0)$, are of the order of 10^{308} with their ratio approximately equal to one), we followed in practice another method. In fact, it can be shown (Brezinski 1977) that the application of the first confluent form of the ε algorithm to the function $S(t)$ is equivalent to solving the system of linear equations

$$\begin{cases} b_0 S(t_0) + b_1 S'(t_0) + \ldots + b_n S^{(n)}(t_0) = c \\ b_0 S'(t_0) + b_1 S''(t_0) + \ldots + b_n S^{(n+1)}(t_0) = 0 \\ \qquad \ldots \\ b_0 S^{(n)}(t_0) + b_1 S^{(n+1)}(t_0) + \ldots + b_n S^{(2n)}(t_0) = 0 \end{cases} \tag{4.3}$$

where $c \neq 0$ is an arbitrary constant, for the unknown b_0. The nth approximant can then be evaluated as

$$\varepsilon_{2n}(t_0) = c/b_0; \tag{4.4}$$

actually, (3.3) is just the Cramer rule expression for the solution of the system (4.3). For comparison we also evaluated the approximants $\varepsilon_{2n}(t_0)$ iteratively, making use of the recurrence relation (Brezinski 1977) between the Hankel functional determinants

$$H_{k+2}^{(n-1)}(t) H_k^{(n+1)}(t) + [H_{k+1}^{(n)}(t)]^2 = H_{k+1}^{(n-1)}(t) H_{k+1}^{(n+1)}(t), \tag{4.5}$$

with the initial conditions

$$H_0^{(n)}(t) = 1, \tag{4.6}$$

$$H_1^{(n)}(t) = S^{(n)}(t), \tag{4.7}$$

for $n = 0, 1, \ldots$. The results obtained with the two procedures are in excellent agreement, to between 22 and 26 significant figures.

In table 1 we report the ground state energy level of the quartic anharmonic oscillator for three values of the coupling constant β; the value t_0 that we chose, taking into account the restrictions imposed by (3.9), is also presented (we recall that $\rho_B = \frac{2}{3}$; see (2.11)). In table 2 the rate of convergence of the approximants $\varepsilon_{2n}(t_0)$ is displayed for two of the above values of β: the successive approximations appear to converge decreasing monotonically in this range of β ($\beta \leq 1$). Obviously, the $\varepsilon_{2n}(t_0)$ converge more and more slowly as β increases because of the finite radius of convergence of the Borel transform series, as we have already mentioned in § 3. We may also remark

Table 1. Ground state energy eigenvalue of the quartic anharmonic oscillator and the value t_0 adopted for the calculation as a function of the coupling constant β. The Borel transform series is as in (2.10).

β	$E_0(\beta)$	t_0
0.01	1.007 373 672 081 382 460 533 843 9	20
0.1	1.065 285 509 543 717 688 8 (7)	2
1	1.392 35 (2)	0.2

Table 2. Rate of convergence of the approximants $\varepsilon_{2n}(t_0)$ as a function of the order n.

n	$\varepsilon_{2n}(t_0)$; $\beta=10^{-2}$	$\varepsilon_{2n}(t_0)$; $\beta=10^{-1}$
1	1.007 373 672 081 574 859 811 756 2	1.066 406 050 317 694 789 73
2	1.007 373 672 081 382 523 263 359 6	1.065 306 142 599 372 735 58
3	1.007 373 672 081 382 460 588 409 1	1.065 286 316 155 262 596 80
4	1.007 373 672 081 382 460 533 929 9	1.065 285 557 270 270 955 37
5	1.007 373 672 081 382 460 533 844 1	1.065 285 513 295 134 890 99
6	1.007 373 672 081 382 460 533 843 9	1.065 285 509 907 925 452 79
—	—	—
10	—	1.065 285 509 543 846 692 53
—	—	—
15	—	1.065 285 509 543 717 729 07
—	—	—
20	—	1.065 285 509 543 717 688 87

that the choice $t_0=0$, which *a priori* may seem to be very appealing because it requires no approximate integration for the evaluation of $S(t_0)$ (3.8), is not convenient in practice. In fact it has already been shown (Marziani 1983) that, with the choice $t_0=0$, the sequence of approximations $\varepsilon_{2n}(t_0)$ to the Borel sum (3.1) coincides with the sequence of $[n, n-1]$ Padé approximants to the perturbation series (2.6). It is known (Graffi and Grecchi 1978) that the Padé approximants do not converge to the true eigenvalues of a generalised anharmonic oscillator (1.1) when $m \geq 4$: therefore the arbitrary parameter t_0 offers a measure of the transition between the convergence and non-convergence of our approximation scheme.

We also applied the second approach outlined in § 3, namely, we approximated the generalised inverse Borel transform (3.14) by means of the first confluent form of the ε algorithm. In this case the Borel transform (3.13) has an infinite radius of convergence, which allows one to choose larger values of t_0 than in the previous case. In table 3 we present our results for the ground state level of $H=p^2+x^2+\beta x^4$, obtained with this second procedure, for various values of β in the intermediate region, employing the first 86 coefficients of the series (2.6); also shown is the value of t_0 which allowed the Borel series (3.13) and its derivatives to be easily summed and, at the same time, to achieve good convergence. The results displayed in table 3 refer to a calculation of 25 approximants $\varepsilon_{2n}(t_0)$; we may add that, unfortunately, in this case the monotony in the convergence pattern is lost.

In order to check our results we compared them with various accurate perturbative calculations of the same eigenvalues, such as those of Hirsbrunner (1982) and Caswell (1979), and with the Padé (Loeffel *et al* 1969, Simon 1970) and Borel–Padé (Graffi

Table 3. Ground state energy eigenvalue of the quartic anharmonic oscillator and the value t_0 adopted for the calculation as a function of the coupling constant β. The Borel transform series is as in (3.13).

β	$E_0(\beta)$	t_0
0.1	1.065 285 509 543 717 688 857 09$\binom{7}{8}$	12
0.2	1.118 292 654 367 039 153$\binom{6}{7}$	9
0.3	1.164 047 157 353 842	8.5
0.4	1.204 810 327 372 501	8
0.5	1.241 854 059 651 514	7.5
0.6	1.275 983 566 342 6$\binom{5}{6}$	7
0.7	1.307 748 651 120$(^{4})$	6.5
0.8	1.337 545 208 149	6
0.9	1.365 669 825 788	5.5
1	1.392 351 641 53$(^{9})$	5
10	2.449 17$(^{8})$	2

et al 1970) results. Our results are in excellent agreement with those obtained in these previous calculations, especially with those of Hirsbrunner (1982) which were probably performed with the same number of significant figures as ours. When the values that we present in table 1 and table 3 are compared with the results of accurate non-perturbative (Biswas *et al* 1973, Banerjee 1978) or variational (Graffi and Grecchi 1973) computations, the situation proves to be less impressive than previously for large values of the coupling constant β. However, in any case good or satisfactory agreement is achieved depending on the value of β, keeping in mind that our approximation scheme is perturbative in nature.

Finally, we applied the same method to the calculation of the ground level of $H = p^2 + x^2 + \beta x^8$, which is already a difficult task in a perturbative approach, due to the strong divergence of the perturbation series coefficients. In this case we employed the Borel transform series (3.16) with the first 43 Rayleigh–Schrödinger coefficients which we re-computed, the first 37 of which are in complete agreement with those calculated by Graffi *et al* (1971). The results we obtained are presented in table 4 and they are in good agreement with the 'exact' ones.

In conclusion, we wish to point out that rather than attempt to obtain the most accurate tables of eigenvalues of the anharmonic oscillators, our aim was to show that the Borel summability method can be made more appealing, also from the computational point of view, by combining it with a rapidly converging extrapolation method. This basic idea should prove useful when dealing with other summability problems in quantum mechanics and quantum field theory.

Table 4. Ground state energy eigenvalue of the octic anharmonic oscillator and the value t_0 adopted for the calculation as a function of the coupling constant β. The Borel transform series is as in (3.16).

β	$E_0(\beta)$	t_0
1×10^{-4}	1.000 646 369 874 074 347	25
1×10^{-3}	1.005 857 514 12	15
1×10^{-2}	1.039 496 7	8
1×10^{-1}	1.168 9$\binom{4}{7}$	4.5

Acknowledgments

Thanks are expressed to Professor L Lovitch for continuously encouraging our work and for reading the manuscript.

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